Ch. 12

Coord. system

distance $\quad |P_1P_2| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

Sphere of rad. r center P_0

$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=r^2$$

Vectors : $\quad \vec{v} = \langle v_1, v_2, v_3 \rangle$

length $\quad |\vec{v}| = \sqrt{v_1^2+v_2^2+v_3^2}$

unit vector in dir of $\vec{v}$: $\quad \dfrac{\vec{v}}{|\vec{v}|}$

Dot product $\quad \vec{u}\cdot\vec{v} = u_1v_1+u_2v_2+u_3v_3$

$$\vec{u}\cdot\vec{v} = \cancel{|u|}|u||v|\cos\theta \rightarrow \cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}$$

$\vec{u}$ and $\vec{v}$ are perpendic. if $\vec{u}\cdot\vec{v}=0$.

Vector proj of $\vec{u}$ onto $\vec{v}$:

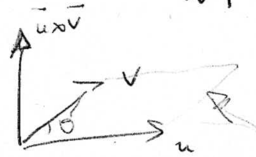
$$\text{proj}_{\vec{v}}\, u = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\vec{v}$$

Scalar component of u in dir of $\vec{v}$:

$$|u|\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|}$$

Cross product

$$\vec{u}\times\vec{v} = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$



$|\vec{u}\times\vec{v}| = |u||v|\sin\theta$ = area of parallelogram

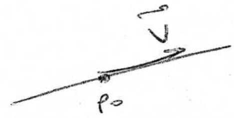
Triple scalar prod

$$(u\times v)\cdot w = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$|(u\times v)\cdot w|$ = Volume of parallelepiped ~~spanned~~ determined by u, v, w

Lines and planes in space

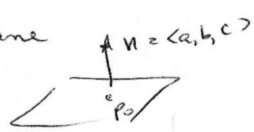
Line $\quad \begin{cases} x = x_0 + v_1 t \\ y = y_0 + v_2 t \\ z = z_0 + v_3 t \end{cases}$



Distance from a point S to a line through P_0 parallel to $\vec{v}$

$$d = \frac{|\vec{P_0S}\times\vec{v}|}{|\vec{v}|}$$

Plane



$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

Distance from a point $P(x_1, y_1, z_1)$ to a plane $ax+by+cz+d=0$

$$D = \frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$$

Angle between planes = acute angle between their normals.

Ch 13 Vector-valued functions.

Curve $\vec{r}(t) = f(t)\,\vec{i} + g(t)\,\vec{j} + h(t)\,\vec{k}$. Differentiation Integration

Velocity $\vec{v} = \vec{r}\,'$

speed $= |\vec{v}|$

acceleration $= \vec{a} = \vec{v}\,' = r''$

Length of a curve

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

$$= \int_a^b |\vec{v}(t)|\,dt$$

Arc length parameter

$$s(t) = \int_{t_0}^{t} |\vec{v}(t)|\,dt \Rightarrow \frac{ds}{dt} = |\vec{v}(t)|$$

Unit tangent vector ; $\vec{T} = \frac{\vec{v}}{|\vec{v}|} = \frac{dr}{ds}$

Kurvature $\kappa = \left|\frac{dT}{ds}\right| = \frac{\left|\frac{dT}{dt}\right|}{|\vec{v}|} = \frac{|a \times v|}{|\vec{v}|^3}$

Principal unit normal $N = \frac{1}{\kappa}\frac{dT}{ds}$

$$= \frac{\frac{dT}{dt}}{\left|\frac{dT}{dt}\right|}$$

Binormal $B = T \times N$

Ch. 14

Functions of several variables

D, R,

open set, closed set, boundary pt, interior point

Bounded, unbounded

Level curve $f(x,y) = C$

Level surface $f(x,y,z) = C$

Limit — Substitution rule → if everything exists
sandwich thm
Two-path test
Continuous functions

Partial deriv

Chain rule — Tree diagram

Directional deriv. $(D_u f)\big|_{P_0} = \nabla f\big|_{P_0} \cdot \bar{u}$
u-unit

$D_u f$ maximal if $u = \frac{\nabla f}{|\nabla f|}$
minimal if $\bar{u} = -\frac{\nabla f}{|\nabla f|}$
$= 0$ if $\bar{u}$ is perpendic. to ∇f.

∇f is perpendicular to level curves of f.

Tangent plane to $f(x,y,z) = c$ at $P_0(x_0, y_0, z_0)$

$$f_x(P_0)(x-x_0) + f_y(P_0)(y-y_0) + f_z(P_0)(z-z_0) = 0$$

Linearization

$$f(x,y) \approx L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

Error in this approximation $|E(x,y)| \leq \frac{1}{2} M\left(|x-x_0| + |y-y_0|\right)^2$
where
$|f_{xx}| \leq M$
$|f_{xy}| \leq M$
$|f_{yy}| \leq M$

Extreme values and saddle points

Crit. point: $f_x = f_y = 0$ or DNE

Local max or min → crit pt.

Second der. test — P - crit. point $D = f_{xx} f_{yy} - f_{xy}^2 \big|_P$

1) $D > 0$ and $f_{xx}\big|_P < 0$ ⇒ loc max
2) $D > 0$ and $f_{xx}\big|_P > 0$ ⇒ loc min
3) $D < 0$ saddle point
4) $D = 0$ inconclusive

Extreme values on closed bounded regions:

Occur either at critical points

or on the boundary.

Lagrange multipliers

Find extreme values of $f(x,y)$ subj to $g(x,y)=0$

Solve $\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y)=0 \end{cases}$

Similarly for $f(x,y,z)$ subj to $g(x,y,z)=0$

$f(x,y,z)$ subj to $g_1(x,y,z)=0$
$g_2(x,y,z)=0$

$$\begin{cases} \nabla f = \lambda \nabla g_1 + \mu \nabla g_2 \\ g_1 = 0 \\ g_2 = 0 \end{cases}$$